Chapter 7.

Polynomials and other functions.

Polynomial functions.

In a linear function the highest power of x is 1: $y = mx^1 + c$

In a quadratic function the highest power of x is 2: $y = ax^2 + bx + c$

Continuing this pattern we have *cubic* functions: $y = ax^3 + bx^2 + cx + d$

quartic functions: $y = ax^4 + bx^3 + cx^2 + dx + e$

etc.

These functions are all part of the larger family of *polynomial* functions.

Polynomial functions have rules of the form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots\ldots + a_2 x^2 + a_1 x + a_0$$

where n is a non-negative integer and a_n, a_{n-1}, a_{n-2}, ... are all numbers, called the **coefficients** of x^n, x^{n-1}, x^{n-2} etc.

The highest power of x is the **order** of the polynomial.

Thus linear functions, $y = mx + c$, are polynomials of order 1,

quadratic functions, $y = ax^2 + bx + c$, are polynomials of order 2.

We will now consider **cubic** functions, i.e. polynomials of order 3.

Cubic functions.

With a = 1 and b = c = d = 0 the general formula for a cubic function:

$$y = ax^3 + bx^2 + cx + d$$

reduces to $y = 1x^3 + 0x^2 + 0x + 0$

i.e. the most basic cubic: $y = x^3$

A table of values and the graph of $y = x^3$ are shown below:

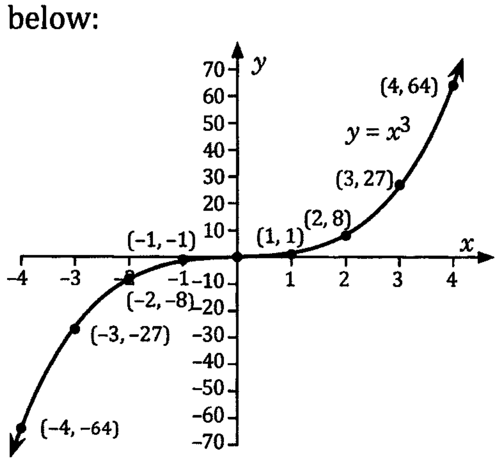
Note that the table of values has a constant <u>third</u> difference – a characteristic of the tables of values for cubic functions.

x	−4	−3	−2	−1	0	1	2	3	4
y	−64	−27	−8	−1	0	1	8	27	64

1st diff 37 19 7 1 1 7 19 37

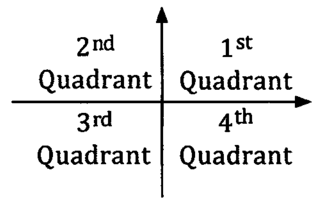
2nd diff −18 −12 −6 0 6 12 18

3rd diff 6 6 6 6 6 6



The graph of $y = x^3$ has certain features that, with thought, we should have expected:

- The cube of any positive number is positive and the cube of a negative number is negative. Thus we would only expect to find $y = x^3$ in the 1st quadrant (see diagram), as that is where both the x and y coordinates are positive, and in the 3rd quadrant, as that is where both x and y coordinates are negative.



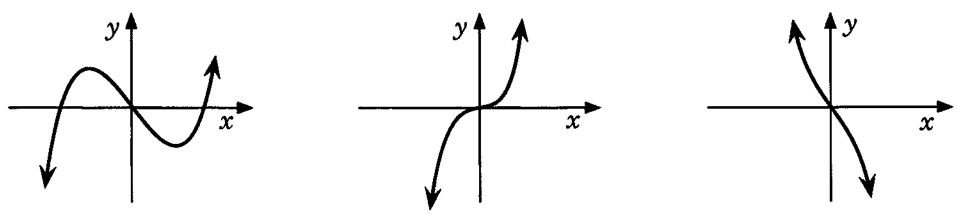
- We would expect the graph to pass through (0, 0).
- As x gets large positively we would expect y to be even larger and positive.
- As x gets large negatively we would expect y to be even larger and negative.
- For every point (x, y) on the graph there will also exist a point $(-x, -y)$, e.g. (2, 8) and (−2, −8), (3, 27) and (−3, −27). I.e. for $f(x) = x^3$, $f(-a) = -f(a)$. This gives the graph its rotational symmetry.

These "expected features" are indeed evident on the graph.

Cubic functions will not all give the same shape graph as that of $f(x) = x^3$ but they all

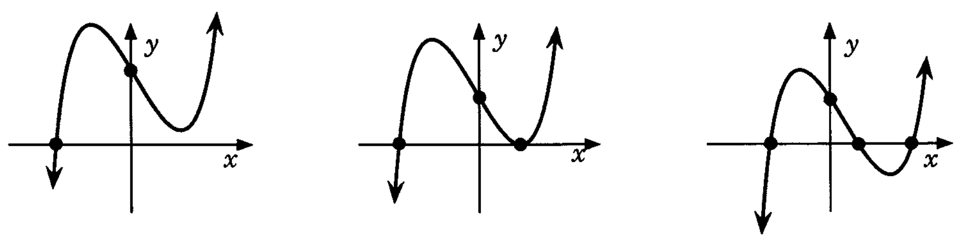
☞ have either two turning points or no turning points,

eg:



$y = x^3 - x$ $\qquad$ $y = x^3$ $\qquad$ $y = -x^3 - x$

and ☞ cut the y-axis once and cut (or touch) the x-axis in 1, 2 or 3 places:



Example 1

Determine the coordinates of the point where the graph of the cubic function

$$y = 2x^3 + 3x^2 + 2x + 14$$

cuts the y-axis.

All points on the y-axis have an x-coordinate of zero.

If $x = 0$

$$\begin{aligned} y &= 2(0)^3 + 3(0)^2 + 2(0) + 14 \\ &= 14 \end{aligned}$$

The given cubic function cuts the y-axis at the point (0, 14).

Cubics in factorised form.

Just as some quadratic functions, $y = ax^2 + bx + c$, may be expressed in the factorised form, $y = a(x - p)(x - q)$,

similarly some cubic functions $y = ax^3 + bx^2 + cx + d$, may be expressed in the factorised form $y = a(x - p)(x - q)(x - r)$

Example 2

Determine the coordinates of the points where the graph of the cubic function

$$y = (x - 1)(2x - 3)(x + 2)$$

cuts (a) the x-axis, (b) the y-axis.

(a) All points on the x-axis have a y-coordinate of zero.

If $y = 0$ $\quad 0 = (x - 1)(2x - 3)(x + 2)$

If three brackets have a product of zero, one of the brackets must equal zero.

i.e. $(x - 1) = 0$ or $(2x - 3) = 0$ or $(x + 2) = 0$

giving $x = 1$ or $2x = 3$ or $x = -2$

i.e. $x = 1{\cdot}5$

The graph cuts the x-axis at (1, 0), (1·5, 0) and (−2, 0).

(b) All points on the y-axis have an x-coordinate of zero.

Given $\quad y = (x - 1)(2x - 3)(x + 2)$

If $x = 0$ $\quad y = (0 - 1)(2 \times 0 - 3)(0 + 2)$

$= (-1)(-3)(2)$

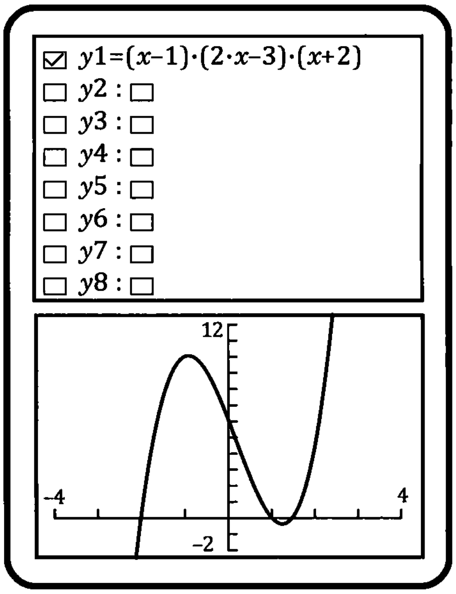
$= 6$

The graph cuts the y-axis at (0, 6).

The graph of

$$y = (x - 1)(2x - 3)(x + 2)$$

is shown on the right.



Investigate.

Use your graphic calculator to view the graphs of cubic functions of the form

$$y = (x - b)(x - c)^2$$

i.e. $\quad y = (x - b)(x - c)(x - c)$

and of the form

$$y = (x - b)^3$$

i.e. $\quad y = (x - b)(x - b)(x - b)$.

What effect does the presence of a "repeated bracket" have on the graph of the function?

Sketching cubic functions.

Nowadays, with the ready access to graphic calculators, the easiest way to view the graph of a cubic function is to display it on a calculator (as in example 4 on the next page). However, if you are required to make a sketch without using a graphic calculator, and provided the cubic is given in factorised form, or in a form that can be readily factorised, for example $y = x^3 - 5x^2 + 6x$, we can determine sufficient information about the cubic for a reasonable sketch to be made, as in examples 3 and 5 that follow. (Or, if not, other points lying on the curve can be determined for other x values.)

Example 3

For the cubic function $y = -2(x + 3)(x - 2)(x - 5)$.

Find the coordinates of any points where the graph of the function cuts

(a) the y-axis, (b) the x-axis.

Describe the behaviour of the y values as the x values become

(c) increasingly large positively, (d) increasingly large negatively.

(e) Hence sketch the function.

(a) All points on the y-axis have an x-coordinate of zero.

If $x = 0$ $\quad y = -2(0 + 3)(0 - 2)(0 - 5)$

$= -60$

The graph cuts the y-axis at $(0, -60)$.

(b) All points on the x-axis have an y-coordinate of zero.

If $y = 0$ $\quad 0 = -2(x + 3)(x - 2)(x - 5)$

i.e. $x + 3 = 0$ or $x - 2 = 0$ or $x - 5 = 0$

The graph cuts the x-axis at $(-3, 0)$, $(2, 0)$ and $(5, 0)$.

(c) As x becomes increasingly large positively (or negatively), its value will dominate each of the three brackets. Thus:

For x large and positive $\quad y = -2(x + 3)(x - 2)(x - 5)$

becomes $\quad y = -2 \times (\text{large +ve}) \times (\text{large +ve}) \times (\text{large +ve})$

= a very large negative number.

As x becomes increasingly large positively, y becomes very large negatively.

I.e.: As $x \to \infty$, $y \to -\infty$.

(d) For x large and –ve $\quad y = -2(x + 3)(x - 2)(x - 5)$

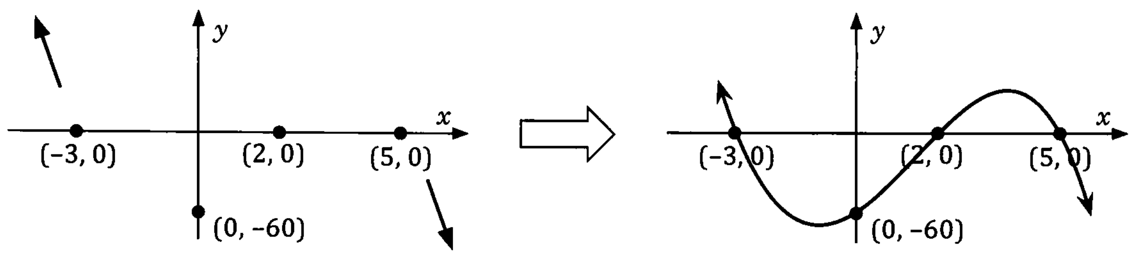
becomes $\quad y = -2 \times (\text{large –ve}) \times (\text{large –ve}) \times (\text{large –ve})$

= a very large positive number.

As x becomes increasingly large negatively, y becomes very large positively.

i.e.: As $x \to -\infty$, $y \to \infty$.

(e) Placing these facts on a graph, below left, allows a sketch to be made, below right.



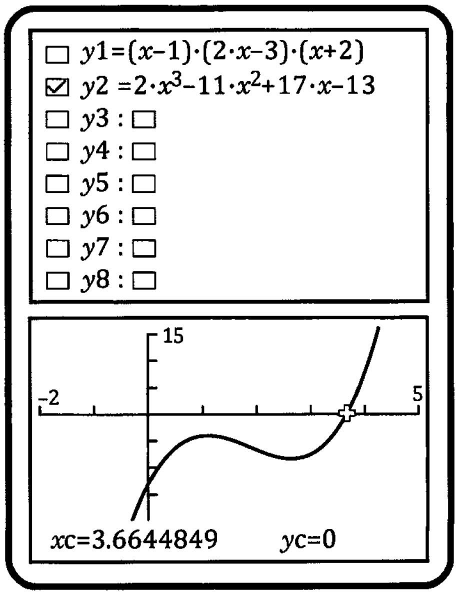
Example 4

Use a graphic calculator to view the graph of the cubic function

$$y = 2x^3 - 11x^2 + 17x - 13.$$

Use your calculator to determine the coordinates of the point(s) where the function cuts the x-axis, rounding any x-coordinates to two decimal places.

A typical calculator display is shown on the right and includes the coordinates of the point where the graph cuts the x-axis.
Hence the coordinates are, to the required accuracy, (3·66, 0).



Example 5

Without the assistance of a calculator produce a sketch of the graph of the cubic function with equation: $y = 2(x+1)(x-5)^2$.

If $y = 0$ $\quad 0 = 2(x+1)(x-5)^2$.

Hence the cubic function cuts the x-axis at the point (−1, 0)
and *touches* the x-axis at the point (5, 0). (The repeated bracket indicating the "touch" of the x-axis as you may have discovered from the investigation on an earlier page.)

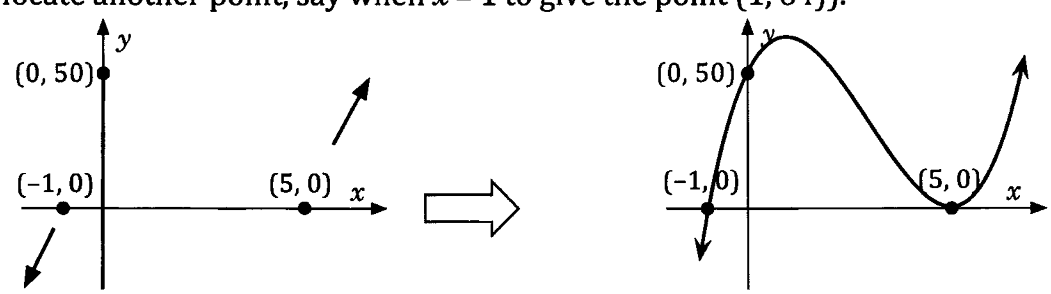
If $x = 0$ $\quad y = 2(0+1)(0-5)^2$.

Hence the cubic function cuts the y-axis at the point (0, 50).

As x gets large positively y gets very large positively. (As $x \to \infty, y \to \infty$.)

As x gets large negatively y gets very large negatively. (As $x \to -\infty, y \to -\infty$.)

Placing these facts on a graph allows a sketch to be made as shown below (or, if still in doubt, locate another point, say when $x = 1$ to give the point (1, 64)).



Exercise 7A

1. Determine the coordinates of the point where the graph of each of the following cubic functions cut the y-axis.

(a)	$y = x^3 + x^2 + x + 1$	(b)	$y = 3x^3 - 5x^2 - 2x - 5$
(c)	$y = x^3 + 8$	(d)	$y = 2x^3 + 3x^2 + 6$
(e)	$y = 2 + 3x + 7x^2 - x^3$	(f)	$y = 5x + 3 + 2x^3$

2. Determine the coordinates of the point(s) where the graph of each of the following cubic functions cut, or perhaps just "touches", the x-axis.

(a)	$y = (x-2)(x-3)(x-4)$	(b)	$y = (x+7)(x-1)(x-5)$
(c)	$y = (2x-5)(x+1)(5x-3)$	(d)	$y = (1-x)(1+x)(x-7)$
(e)	$y = x(4x-1)(2x-7)$	(f)	$y = (x+1)^2(x-5)$
(g)	$y = x^3 - 9x$	(h)	$y = x^3 + 2x^2 - 15x$

3. Use a graphic calculator to view the graph of the cubic function

$$y = 2x^3 - 2x^2 - 3x - 5.$$

Use your calculator to determine the coordinates of the point(s) where the function cuts the x-axis, rounding any x-coordinate(s) to two decimal places.

4. (a) Given that $x^3 + 5x^2 - 12x - 36 = (x+2)(x-3)(x-k)$ find k.
   (b) Find the coordinates of any point(s) where $y = x^3 + 5x^2 - 12x - 36$ cuts the x-axis.

5. If $f(x) = x^3 - 6x^2 - x + 6$ determine (a) f(−1) (b) f(1) (c) f(2) (d) f(6). Hence factorise $x^3 - 6x^2 - x + 6$.

6. If $f(x) = x^3 - 10x^2 + 31x - 30$ determine (a) f(1) (b) f(2) (c) f(3). Hence factorise $x^3 - 10x^2 + 31x - 30$.

7. Given that $3x^3 - 14x^2 - 7x + 10 = (3x-2)(ax^2 + bx + c)$:
   (a) Determine the value of a and the value of c by inspection.
   (b) With your answers from part (a) in place expand $(3x-2)(ax^2 + bx + c)$ and hence determine b.
   (c) Find the coordinates of any x-axis intercepts of the graph of

$$y = 3x^3 - 14x^2 - 7x + 10.$$

8. By determining
   - the coordinates of any points where the function cuts, or perhaps just touches the axes,

   and by considering
   - the behaviour of the function as x gets large positively and negatively,

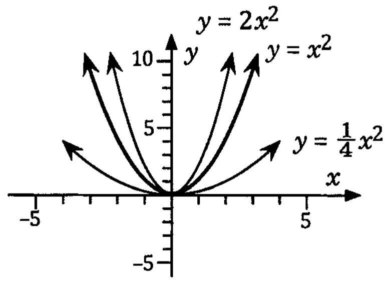
   produce sketches of each of the following cubic functions.
   Then check the reasonableness of each sketch by viewing the graph of the function on a graphic calculator.

(a)	$y = (x+2)(x-2)(x-5)$	(b)	$y = (x+4)(x+1)(x-5)$
(c)	$y = 2(x+4)(x+1)(x-5)$	(d)	$y = x(3-x)(x-7)$
(e)	$y = (x-1)(x-3)^2$	(f)	$y = (x-2)^3$

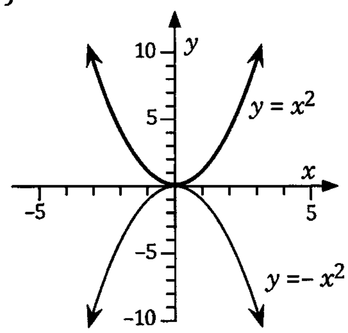
Transformations.

From chapter 5, *Quadratic functions*, you should be familiar with the following ideas:

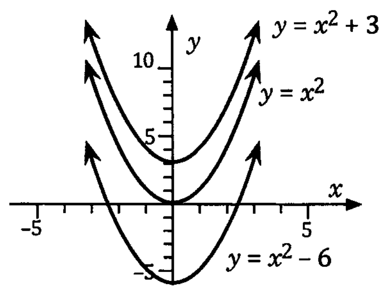
- Altering the "a" in $y = ax^2$ stretches, *dilates*, the graph vertically. (Points on the x-axis are unmoved.)



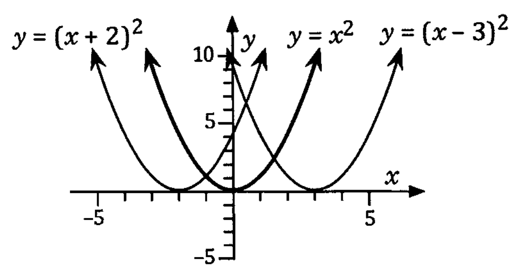
- The graphs of $y = -x^2$, $y = -2x^2$ etc. are simply those of $y = x^2$, $y = 2x^2$ etc., *reflected* in the x-axis.



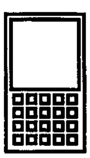
- Altering the "c" in $y = x^2 + c$ *translates* the graph vertically.



- Altering the "b" in $y = (x - b)^2$ *translates* the graph horizontally.



Hence the graph of $y = a(x - b)^2 + c$ is that of $y = x^2$ stretched vertically, scale factor a, (and reflected in the x-axis if a is negative), translated b units right and then c units up.



Do such ideas also apply to cubic functions?

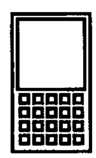
I.e. Does altering the "a" in $y = ax^3$ stretch (dilate) the graph vertically?

Are the graphs of $y = -x^3$, $y = -2x^3$ etc. simply those of $y = x^3$, $y = 2x^3$ etc., reflected in the x axis?

Does altering the "c" in $y = x^3 + c$ translate the graph vertically?

Does altering the "b" in $y = (x - b)^3$ translate the graph horizontally

Investigate



So how does the graph of $y = a(x - b)^3 + c$ compare to the graph of $y = x^3$?

What will the graphs of $y = \frac{1}{x}$ and $y = \sqrt{x}$ look like?

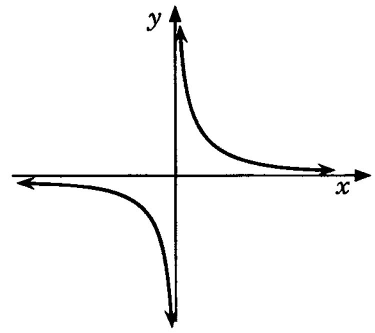
$$y = \frac{1}{x}$$

The *Preliminary work* section at the beginning of this book reminded us that functions of the form

$$y = \frac{k}{x}$$

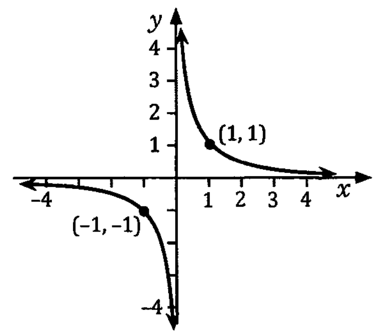
☞ have graphs with the characteristic shape shown on the right (reflected in the y-axis if k is negative).

☞ describe situations in which the two variables are inversely proportional.



☞ have tables of values for which paired values have a common product.

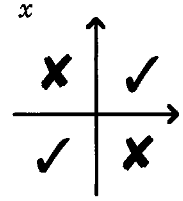
Hence $y = \frac{1}{x}$ has this characteristic shape:



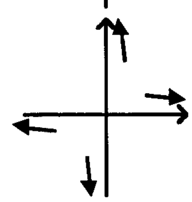
Again, with thought, this shape is exactly as we should have expected for $y = \frac{1}{x}$ because:

☞ When x is positive y will be positive and when x is negative y will be negative.

Thus the graph only exists where x and y are of the same sign:



☞ If x is large, y must be small and if x is small y must be large:



☞ The function does not exist for $x = 0$ and there are no values of x for which y equals 0. Thus the function does not cut either axis.

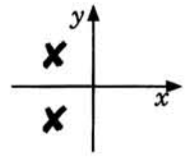
Note

- The x axis is said to be a horizontal **asymptote** to the curve and the y axis is a vertical **asymptote**. These are lines that the curve gets closer and closer to without ever quite touching.
- For every point (x, y) on the graph there will also exist a point $(-x, -y)$, for example (1, 1) and (−1, −1), (2, 0·5) and (−2, −0·5). I.e. $f(-a) = -f(a)$. This gives the graph its rotational symmetry.
- The graph is said to be **hyperbolic**.

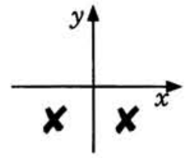
$y = \sqrt{x}$

Again let us think about some of the characteristics we would expect the graph of $y = \sqrt{x}$ to possess.

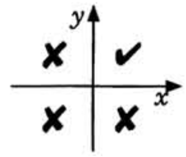
☞ In our system of real numbers we cannot determine the square root of a negative number. Hence the graph does not exist for negative values of x.



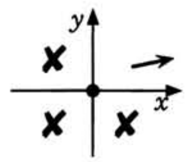
☞ There are also no values of x for which $\sqrt{x}$ will be negative. Hence the graph does not exist for negative values of y.



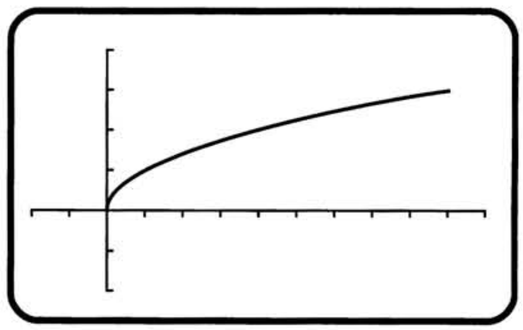
Putting the two previous ideas together we have $y = \sqrt{x}$ only existing where x and y both take non negative values.



☞ We would expect the graph of $y = \sqrt{x}$ to include the point (0, 0) and as x gets large positively y would also get large positively but at a slower rate.



These "expected features" are indeed evident on the graph of $y = \sqrt{x}$:



How does the graph of $y = \dfrac{a}{x - b} + c$ compare to the graph of $y = \dfrac{1}{x}$?

How does the graph of $y = a\sqrt{x - b} + c$ compare to the graph of $y = \sqrt{x}$?

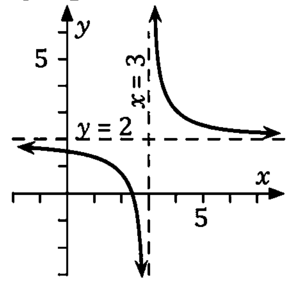
	**Investigate and write a summary of your findings.**	

Vocabulary.

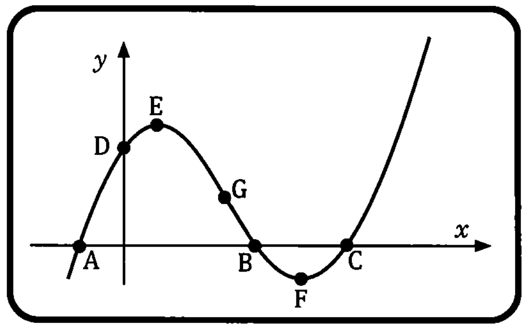
Some vocabulary that we will commonly encounter when discussing the shape and notable features of the graphs of functions is explained below. Some of these terms we are already familiar with from this and previous chapters of this text.

For example, an earlier page considered a curve that got closer and closer to some lines without ever quite touching the lines and referred to these lines as **asymptotes**

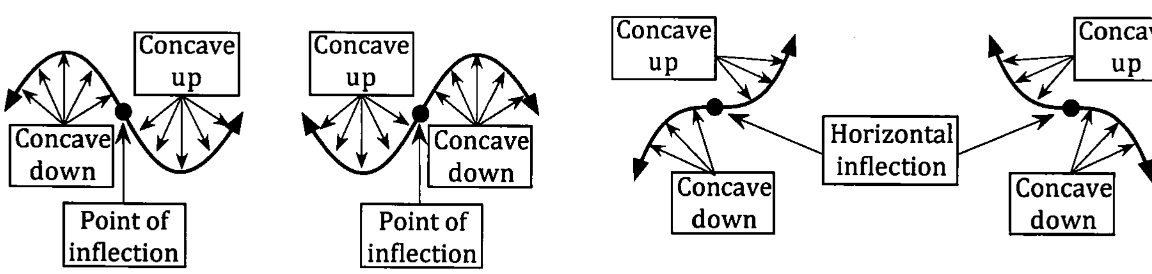
In the diagram on the right the lines $x = 3$ and $y = 2$ are asymptotes to the curve



The graph shown on the right does not appear to have any asymptotes but some other noteworthy features are:



- The ***x*-axis intercepts.**
  Points A, B and C in the diagram.
- The ***y*-axis intercepts.**
  Point D in the diagram.
- Any **turning points** the graph has.
  Points E and F in the diagram.
  E is a **maximum turning point** and **F is a minimum turning point.**
- If $y = f(x)$ is shaped ∩ (or part of ∩) we say that it is **concave down.**
  The graph shown above appears to be concave down to the left of point G.
- If $y = f(x)$ is shaped ∪ (or part of ∪) we say that it is **concave up.**
  The graph shown above appears to be concave up to the right of point G.
- The points on a curve where it changes from being concave down to concave up, or from concave up to concave down, are called **points of inflection.** Point G in the above diagram is a point of inflection.
  If, at a point of inflection, the graph is momentarily horizontal then the point is a point of **horizontal inflection.**



Also remember:

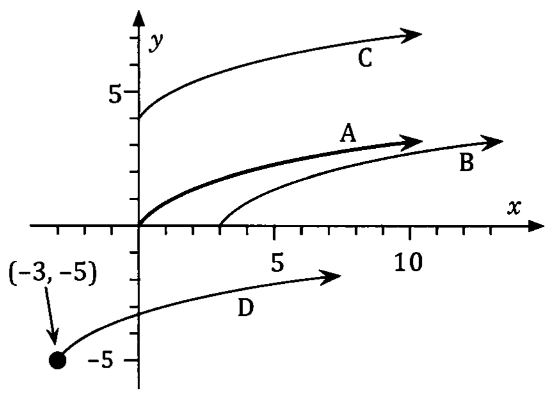
- Some graphs possess **symmetry**.
  Eg $y = x^2$ has *line symmetry*, $y = x^3$ has *rotational symmetry*.

- Some functions have regions on the graph where the function is undefined. The natural domain is then not the entire set of real numbers. For example $y = \sqrt{x}$ is undefined for $x < 0$. The natural **domain** of $y = \sqrt{x}$ is $\{x \in \mathbb{R}: x \geq 0\}$.

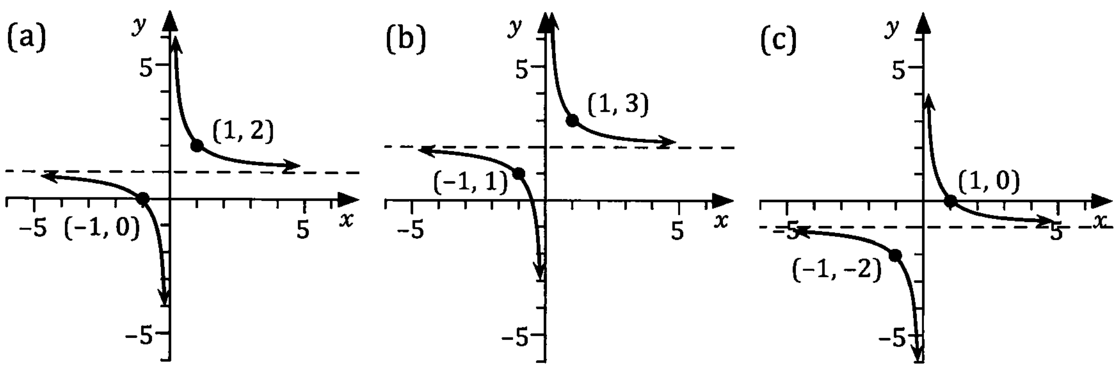
- Some functions do not output all of the real numbers. For example $y = x^2 + 1$ will not output any numbers less that 1. Hence for $x \in \mathbb{R}$ the function $y = x^2 + 1$ has a **range** of $\{y \in \mathbb{R}: y \geq 1\}$.

Exercise 7B

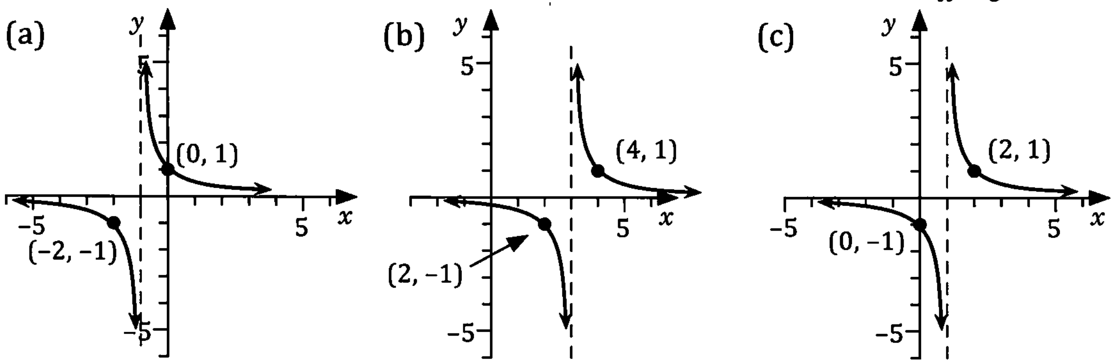
1. Graph A shown in bold on the right has equation $y = \sqrt{x}$.
   Graphs B, C and D are all translations of graph A.
   Write down the equations of B, C and D.



2. Find the equation of each graph below, given each is of the form $y = \frac{1}{x} + c$.



3. Find the equation of each graph below, given each is of the form $y = \frac{1}{x + c}$.



4. Describe how the graph of $y = x^3 + 1$ compares to that of $y = x^3$.

5. Describe how the graph of $y = \frac{1}{x-1}$ compares to that of $y = \frac{1}{x}$.

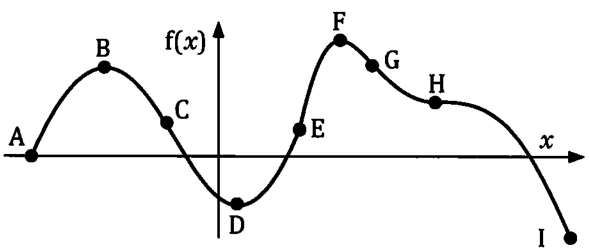
6. Describe how the graph of $y = 2\sqrt{x}$ compares to that of $y = \sqrt{x}$.

7. Describe how the graph of $y = (x-3)^2$ compares to that of $y = (x+4)^2$.

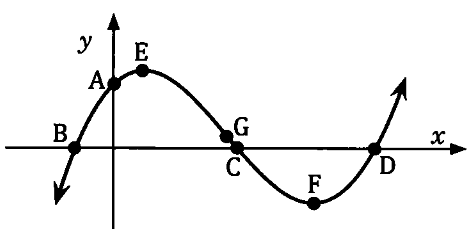
8. Describe how the graph of $y = \sqrt{x-2} + 1$ compares to that of $y = \sqrt{x}$.

9. Describe how the graph of $y = \frac{3}{x-1}$ compares to that of $y = \frac{1}{x}$.

10. For the function on the right state which of the points A to I are:
    (a) The 2 maximum turning points
    (b) The minimum turning point.
    (c) The four points of inflection.
    (d) The one point of horizontal inflection.
    (e) Between which points is the function concave up?
    (f) Between which points is the function concave down?



11. The graph sketched on the right is that of
$$y = x^3 - 9x^2 + 15x + 10$$
By viewing the graph of the function on a graphic calculator determine the coordinates of points A to G which are all the intercepts with the axes, the turning points and the point of inflection (which in this case is halfway between the two turning points).



12. According to Boyle's law the pressure of a fixed mass of gas, kept at a constant temperature, is inversely proportional to the volume of the gas.

i.e. $$P = \frac{k}{V}$$

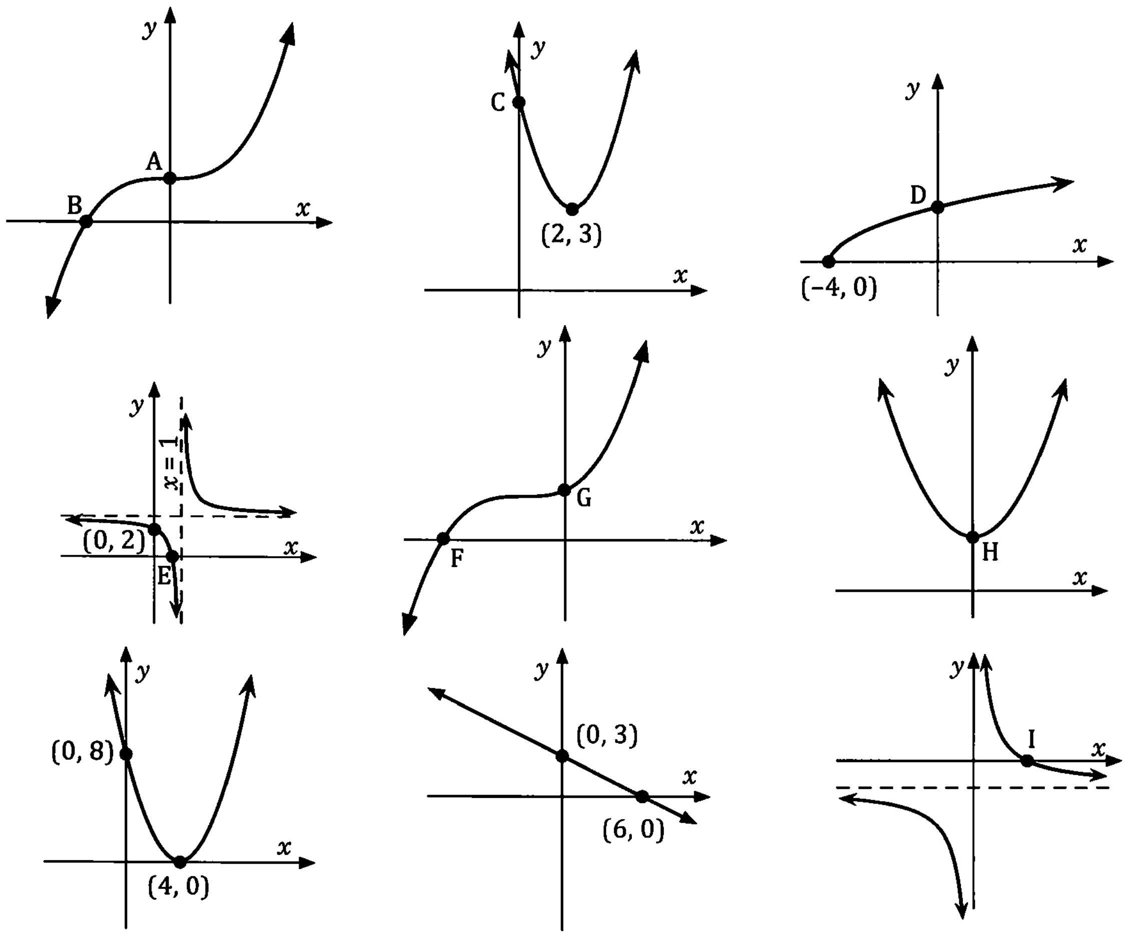
Let us suppose that for a particular, non zero, mass of a gas the pressure, P units of pressure, and the volume, V cm^3, are such that

$$P = \frac{400}{V}$$

(a) Find V when P = 40. (b) Find V when P = 20.
(c) What would be a suitable domain for V?

13. The nine equations given in the box below are for the nine graphs shown below.
Determine the coordinates of points A, B, C, ... I
and the values of a, b, c, ... i .

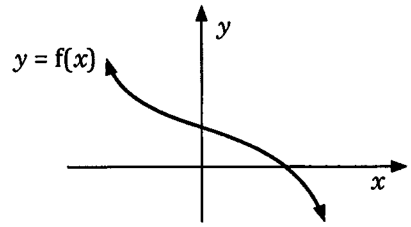
$y = x^3 + 8$	$y = \sqrt{x + a}$	$y = x^2 + 4$
$y = (x + 1)^3 + 8$	$y = \frac{8}{x} - 2$	$y = b(x - c)^2$
$y = (x - d)^2 + e$	$y = \frac{1}{x - f} + g$	$y = hx + i$



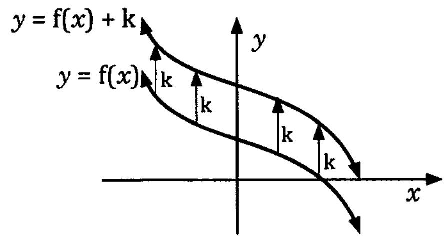
Transformations of the general function $y = f(x)$.

Given the graph of $y = x^2$, or $y = x^3$, or $y = \sqrt{x}$ or $y = \frac{1}{x}$ we have seen that "adding k to the right hand side" of these equations moves the original graph up k units. This, and other transformations given in terms of a more general function $y = f(x)$ are stated below. Note especially "replacing x by $-x$" and "replacing x by ax", because those transformations have not been encountered so far in this text.

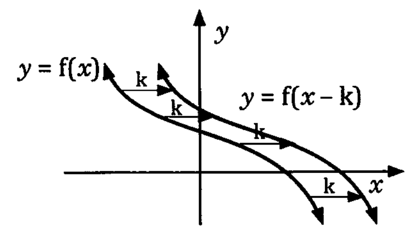
To allow us to illustrate the various transformations let us suppose that some function $y = f(x)$ has the graph shown on the right. (There is no significance in the shape chosen. It could be the graph of any function.)



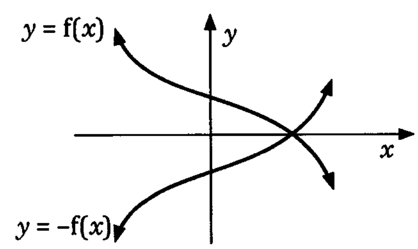
"Adding k to the right hand side."
The graph of $y = f(x) + k$ will be that of $y = f(x)$ translated k units vertically upwards.
Thus if k is negative the translation will be vertically downwards.



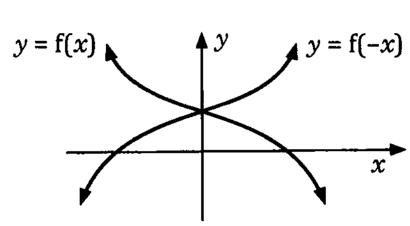
"Replacing x by $(x - k)$."
The graph of $y = f(x - k)$ will be that of $y = f(x)$ translated k units to the right.
Thus if k is negative the translation will be to the left.



"Multiplying the right hand side by −1."
The graph of $y = -f(x)$ will be that of $y = f(x)$ reflected in the x-axis.

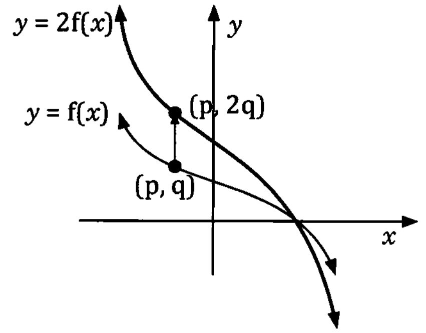


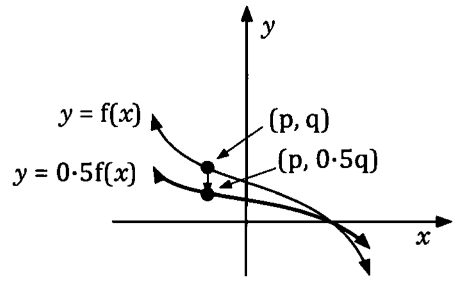
"Replacing x by $-x$."
The graph of $y = f(-x)$ will be that of $y = f(x)$ reflected in the y-axis.



"Multiplying the right hand side by a."
The graph of $y = af(x)$ will be that of $y = f(x)$ dilated parallel to the y-axis with scale factor a. A point that is q units above the x-axis will be moved vertically to a point that is aq units above the x-axis. Points on the x-axis will not move.
If $a > 1$ the effect will be to stretch $y = f(x)$ vertically and if $0 < a < 1$ the effect will be to compress $y = f(x)$ vertically.

Below left shows the situation for a = 2 and below right shows a = 0·5.



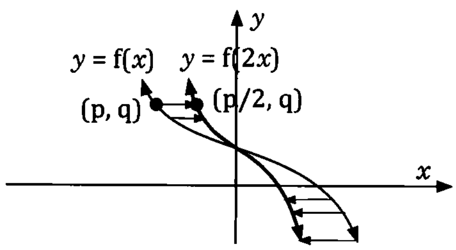


"Replacing x by ax."

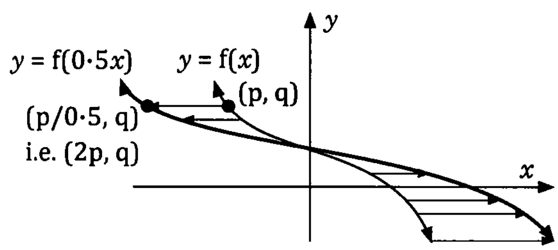
The graph of $y = f(ax)$ will be that of $y = f(x)$ dilated parallel to the x-axis with scale factor $\frac{1}{a}$. A point that is p units from the y-axis will be moved horizontally to a point that is $\frac{p}{a}$ units from the y-axis. Points on the y-axis will not move.

If $a > 1$ the effect will be to compress $y = f(x)$ horizontally and if $0 < a < 1$ the effect will be to stretch $y = f(x)$ horizontally.

Below left shows the situation for a = 2 and below right shows a = 0·5.



For "half the x value" f(2x) will output the same value as f(x) would.



We now need "twice the x value" for f(0·5x) to output the same value as f(x) would.

Using a viewing window of –6 to 6 on the x-axis and –8 to 8 on the y-axis display the graph of $y = x^3 + 2x^2 - x + 3$ on your graphic calculator.

For each of the equations I to VI given below:

- Predict what transformation of the graph of $y = x^3 + 2x^2 - x + 3$ will give the graph of the given equation.
- Display the graph of the given equation on your graphic calculator, along with that of $y = x^3 + 2x^2 - x + 3$, to test your prediction.

I: $y = -(x^3 + 2x^2 - x + 3)$

II: $y = (x-2)^3 + 2(x-2)^2 - (x-2) + 3$

III: $y = (-x)^3 + 2(-x)^2 - (-x) + 3$

IV: $y = 0{\cdot}5(x^3 + 2x^2 - x + 3)$

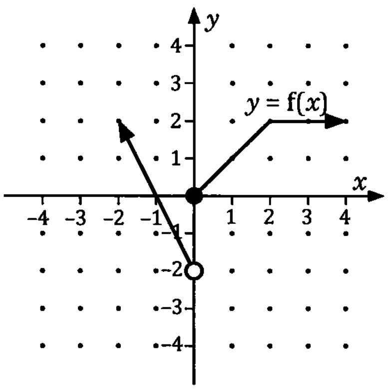
V: $y = (0{\cdot}5x)^3 + 2(0{\cdot}5x)^2 - (0{\cdot}5x) + 3$

VI: $y = x^3 + 2x^2 - x - 2$

Example 6

The graph of $y = f(x)$ is shown on the right.

Note: The "filled" and "empty" circles indicate where the function <u>is</u> (filled circle) and <u>is not</u> (empty circle). Thus f(0) = 0, not −2.



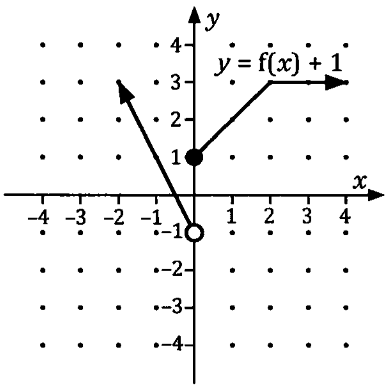
Draw the graph of each of the following.

(a) $y = f(x) + 1$ (b) $y = f(x + 2)$

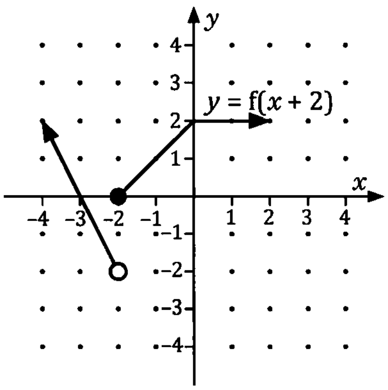
(c) $y = 2f(x)$ (d) $y = f(0{\cdot}5x)$

(e) $y = f(2x)$ (f) $y = -f(x)$

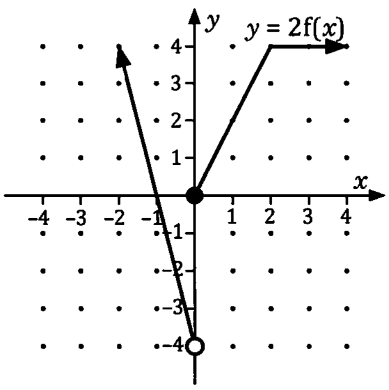
(a) To go from $y = f(x)$
to $y = f(x) + 1$
involves adding 1 to the right hand side.
Thus the graph of $y = f(x) + 1$
will be that of $y = f(x)$
translated vertically upwards 1 unit.



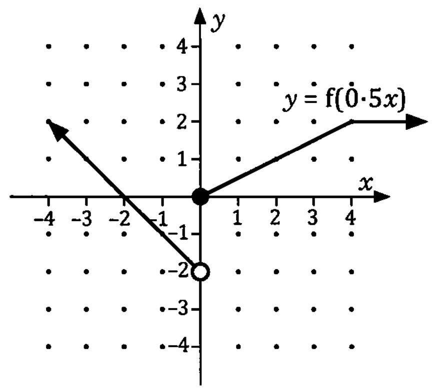
(b) To go from $y = f(x)$
to $y = f(x + 2)$
involves replacing x by $x + 2$.
Thus the graph of $y = f(x + 2)$
will be that of $y = f(x)$
translated 2 units to the left.



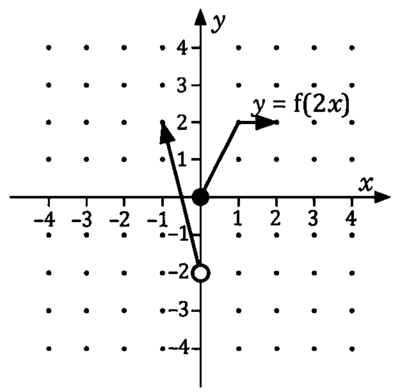
(c) To go from $y = f(x)$
to $y = 2f(x)$
involves multiplying the right hand side by 2.
Thus the graph of $y = 2f(x)$
will be that of $y = f(x)$
dilated parallel to the y-axis, scale factor 2.



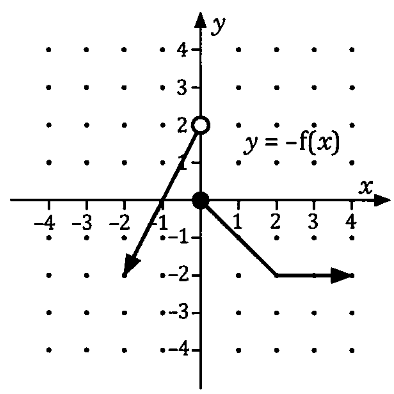
(d) To go from $y = f(x)$
to $y = f(0{\cdot}5x)$
involves replacing x by $0{\cdot}5x$.
Thus the graph of $y = f(0{\cdot}5x)$
will be that of $y = f(x)$
dilated parallel to the x-axis, scale factor $\frac{1}{0{\cdot}5}$
$= 2$.



(e) To go from $y = f(x)$
to $y = f(2x)$
involves replacing x by $2x$.
Thus the graph of $y = f(2x)$
will be that of $y = f(x)$
dilated parallel to the x-axis, scale factor $\frac{1}{2}$
$= 0{\cdot}5$.



(f) To go from $y = f(x)$
to $y = -f(x)$
involves multiplying the right hand side by -1.
Thus the graph of $y = -f(x)$
will be that of $y = f(x)$
reflected in the x-axis.

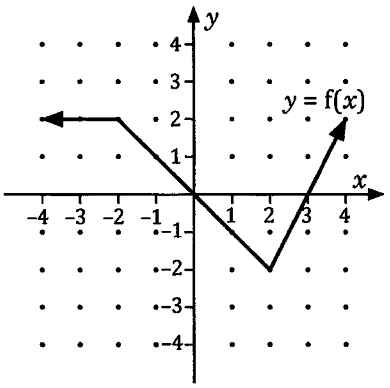


Exercise 7C

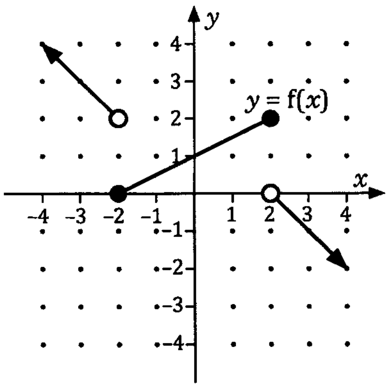
1. Describe how the graphs of each of the following can be obtained by transforming the graph of $y = f(x)$.
   (a) $y = -f(x)$ (b) $y = f(4x)$ (c) $y = 4f(x)$
2. Describe how the graphs of each of the following can be obtained by transforming the graph of $y = x^2 + 3x$.
   (a) $y = -x^2 - 3x$ (b) $y = x^2 + 3x - 5$ (c) $y = \frac{x^2}{4} + \frac{3x}{2}$
3. Describe how the graphs of each of the following can be obtained by transforming the graph of $y = x^2$.
   (a) $y = (x - 3)^2$ (b) $y = 3x^2$ (c) $y = (3x)^2$

4. The graph of $y = \mathrm{f}(x)$ is shown on the right.
Draw the graph of each of the following.

(a) $y = \mathrm{f}(x - 2)$ (b) $y = \mathrm{f}(x) + 2$
(c) $y = 2\mathrm{f}(x)$ (d) $y = \mathrm{f}(2x)$
(e) $y = -\mathrm{f}(x)$ (f) $y = \mathrm{f}(-x)$

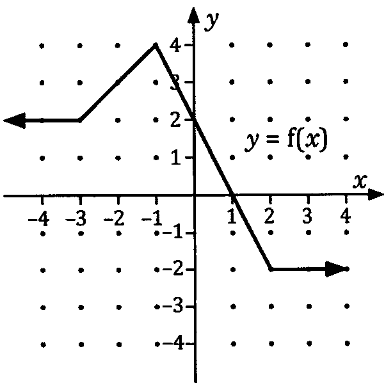


5. The graph of $y = \mathrm{f}(x)$ is shown on the right.
Find
(a) f(0) i.e. the value of y when $x = 0$.
(b) f(1) i.e. the value of y when $x = 1$.
(c) f(2)
(d) f(−3)

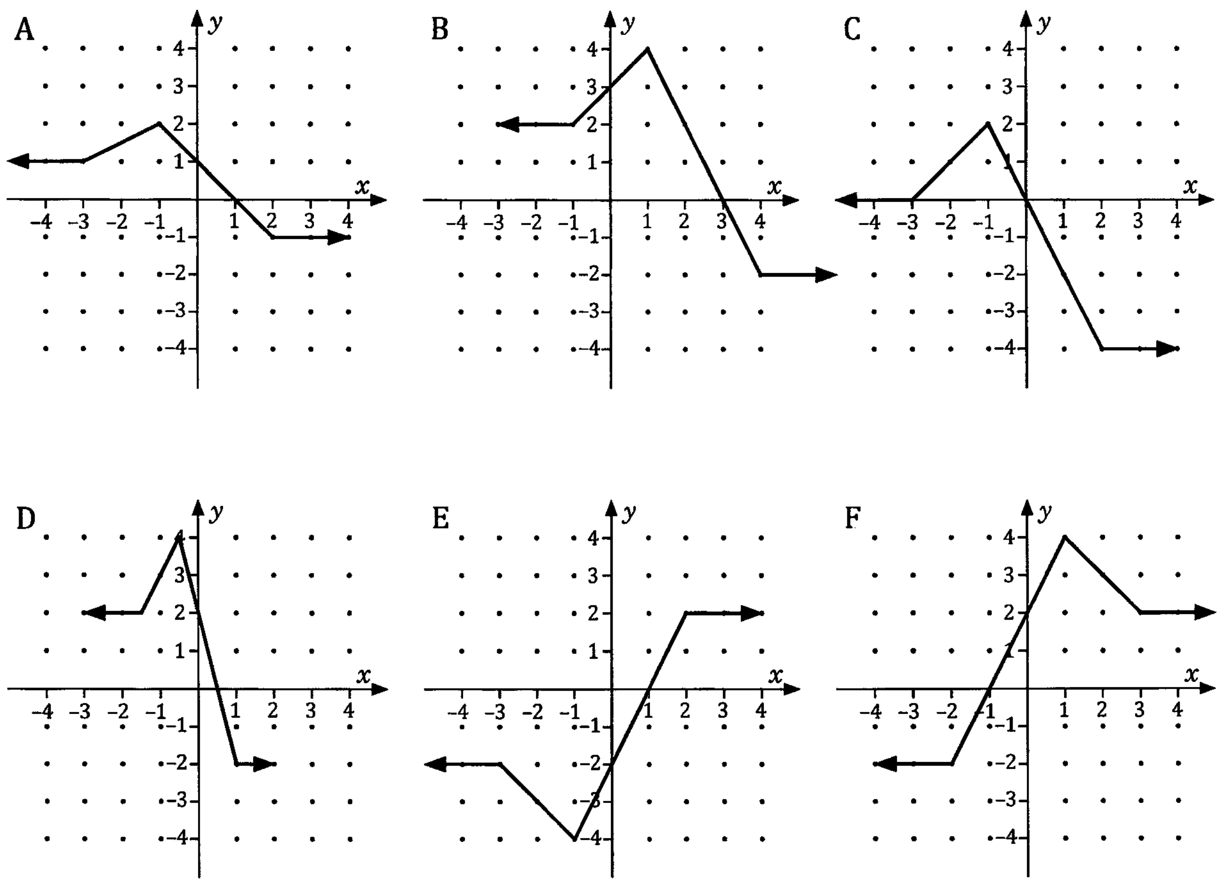


Draw the graph of each of the following.
(e) $y = \mathrm{f}(x + 1)$
(f) $y = \mathrm{f}(-x)$
(g) $y = \mathrm{f}(2x)$
(h) $y = \mathrm{f}(0{\cdot}5x)$
(i) $y = 0{\cdot}5\,\mathrm{f}(x)$
(j) Use your part (b) answer and your part (e) graph to confirm that f(1) = f(0 + 1).
(k) Use your part (c) answer and your part (g) graph to confirm that f(2) = f(2 × 1).

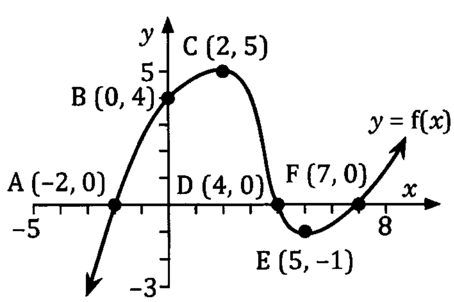
6. The graph of $y = \mathrm{f}(x)$ is as shown on the right. Choose the function from the "functions box" corresponding to each of the graphs A to F shown on the next page.



Functions Box			
I	$y = -\mathrm{f}(x)$	II	$y = \mathrm{f}(-x)$
III	$y = 0{\cdot}5\,\mathrm{f}(x)$	IV	$y = \mathrm{f}(0{\cdot}5x)$
V	$y = 2\,\mathrm{f}(x)$	VI	$y = \mathrm{f}(2x)$
VII	$y = \mathrm{f}(x) + 2$	VIII	$y = \mathrm{f}(x + 2)$
IX	$y = \mathrm{f}(x) - 2$	X	$y = \mathrm{f}(x - 2)$



7. The graph of $y = f(x)$ shown on the right, cuts the x-axis at A(−2, 0), D(4, 0) and F(7, 0), cuts the y-axis at B(0, 4), has a maximum turning point at C(2, 5) and a minimum turning point at E(5, −1).



Find the coordinates of the points where

(a) $y = f(x - 3)$ cuts the x-axis,

(b) $y = f(2x)$ cuts the x-axis,

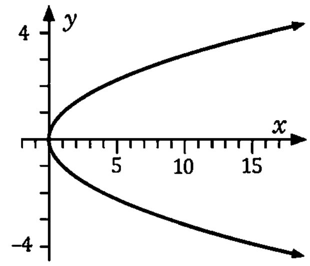
(c) $y = -f(x)$ cuts the x-axis

(d) $y = f(-x)$ cuts the x-axis,

(e) $y = f(x) + 3$ has its maximum turning point,

(f) $y = -f(x)$ has its maximum turning point.

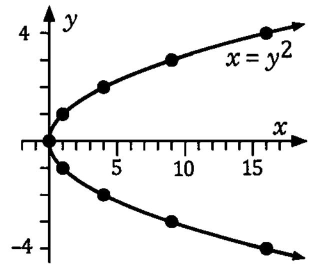
Two relationships that are not functions.



Consider the graph shown on the right.

The graph looks certainly looks parabolic in shape but is it a *quadratic function*?

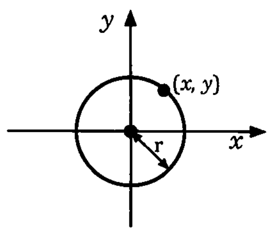
Recalling from chapter 3 the requirement that to be the graph of a function the graph must pass the *vertical line test*, the graph shown cannot be that of a function.
However the graph *is* parabolic in nature and its similarity with the graph of $y = x^2$ is because if we switch the x and y in $y = x^2$, to obtain $x = y^2$ (or $y^2 = x$), we have the equation of the given graph.



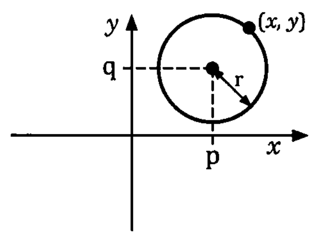
y	−4	−3	−2	−1	0	1	2	3	4
x	16	9	4	1	0	1	4	9	16

The graph of the relationship $y^2 = x$ is parabolic and has the x-axis as its line of symmetry.

Now consider the graphs shown below.



Circle centre (0, 0), radius r.



Circle centre (p, q), radius r.

Again each is <u>not</u> a function because for some x values there exists more than one y value. I.e. each relationship is "one-to-many". Each fails the vertical line test and therefore neither relationship is a function. However, as with $y^2 = x$ above, we can still determine a rule for each relationship.
Remember that the rule for a relationship is like the "membership ticket" for the relationship – all points (x, y) lying on the graph of the relationship will "fit" the rule and all points not on the graph of the relationship will not fit the rule. For the above two relationships we can use the fact that in each case, all points lying on the circles must be a distance r from the centre of the circle (and any point that is not a distance of r from the centre will not lie on the circle) to determine a rule for each relationship. Thus, using the Pythagorean rule for right triangles, we have the rules:

$x^2 + y^2 = r^2$ and $(x - p)^2 + (y - q)^2 = r^2$

for a circle centre (0, 0) and radius r.

for a circle centre (p, q) and radius r.

If we expand $(x - p)^2 + (y - q)^2 = r^2$

we obtain $x^2 - 2px + p^2 + y^2 - 2qy + q^2 = r^2$

i.e. $x^2 + y^2 - 2px - 2qy = r^2 - p^2 - q^2$

i.e. $x^2 + y^2 - 2px - 2qy = \text{(a constant)}$

Notice that in this expanded form the Cartesian equation of a circle is characterised by:

- the coefficient of x^2 being the same as the coefficient of y^2,
- the only terms are those of x^2, y^2, x, y and a constant (and of these any two of the last three could be zero).

Example 7

Find (a) the equation of the circle centre (3, −1) and radius 4.

(b) the centre and radius of the circle with equation:

$$x^2 + y^2 + 6y = 10x$$

(a) The equation of a circle centre (p, q) and radius r is $(x - p)^2 + (y - q)^2 = r^2$

∴ The equation of a circle centre (3, −1) and radius 4 is $(x - 3)^2 + (y + 1)^2 = 16$

(b) We need to rearrange the given equation to the form $(x - p)^2 + (y - q)^2 = r^2$

$$x^2 + y^2 + 6y = 10x$$

$$x^2 - 10x + \ldots\ldots + y^2 + 6y + \ldots\ldots = 0 \quad \leftarrow \text{create gaps}$$

$$x^2 - 10x + 25 + y^2 + 6y + 9 = 0 + 25 + 9 \quad \leftarrow \text{complete the squares}$$

$$(x - 5)^2 + (y + 3)^2 = 34$$

Comparing with $(x - p)^2 + (y - q)^2 = r^2$ we see that the given circle has its centre at (5, −3) and a radius of $\sqrt{34}$.

Exercise 7D

1. Which of the following equations represent circles?

   A: $x^2 + y^2 - 2x + 4y = 6$ B: $2x^2 + y^2 - 3x + 8y + 10 = 0$

   C: $x^2 + y^2 = 6$ D: $x^2 + y^2 + 8x = 10$

   E: $x^2 - y^2 + 2x + 10y = 100$ F: $x^2 + 6xy + y^2 + 15y = 20$

2. Find the equation of a circle centre (0, 0) and radius 10 units.
   If each of the following points lie on this circle determine a, b, c and d given that a and b are positive and c and d are negative.
   Point A (−6, a). Point B (3, b). Point C (0, c). Point D (d, 5).

3. Find the equation of each of the following circles, giving your answers in the form
   $$(x - p)^2 + (y - q)^2 = c$$
   (a) Centre (2, −3) and radius 5. (b) Centre (3, 2) and radius 7.
   (c) Centre (−10, 2) and radius $3\sqrt{5}$. (d) Centre (−1, −1) and radius 6.

4. Find the equation of each of the following circles, giving your answers in the form
$$x^2 + y^2 + dx + ey = c$$
   (a) Centre (3, 5) and radius 5.
   (b) Centre (−2, 1) and radius $\sqrt{7}$.
   (c) Centre (−3, −1) and radius 2.
   (d) Centre (3, 8) and radius $2\sqrt{7}$.

5. Find the radius and the coordinates of the centre of each of the following circles.
   (a) $x^2 + y^2 = 25$
   (b) $25x^2 + 25y^2 = 9$
   (c) $(x - 3)^2 + (y + 4)^2 = 25$
   (d) $(x + 7)^2 + (y - 1)^2 = 100$
   (e) $x^2 + y^2 - 6x + 4y + 4 = 0$
   (f) $x^2 + y^2 + 2x - 6y = 15$
   (g) $x^2 + y^2 + 2x = 14y + 50$
   (h) $x^2 + 10x + y^2 = 151 + 14y$
   (i) $x^2 + y^2 = 20x + 10y + 19$
   (j) $2x^2 - 2x + 2y^2 + 10y = -5$

6. Find the distance between the centres of the two circles given below:
   $(x - 3)^2 + (y - 7)^2 = 36$ $\qquad$ $(x - 2)^2 + (y - 9)^2 = 49$

7. The circle $(x - 3)^2 + (y + 4)^2 = 25$ has its centre at point A.
   The circle $(x - 2)^2 + (y - 7)^2 = 9$ has its centre at point B.
   Find the equation of the straight line through A and B.

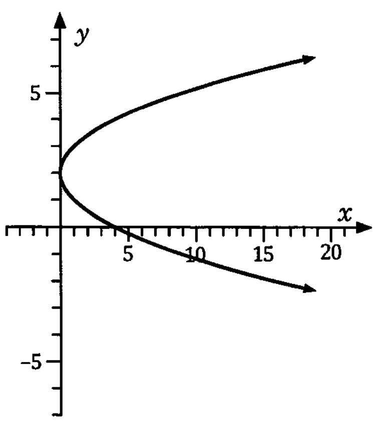
Note: Statements like "adding 4 to the right hand side of the equation will translate the graph up 4 units" could be made for functions with equations of the form $y = f(x)$. However **questions 8, 9 and 10** that follow do <u>not</u> involve functions, and the equations are not in the form $y = f(x)$, so think carefully.

8. The circle $(x + 1)^2 + (y - 7)^2 = 36$ is moved right 4 units and down 3 units. What will be the equation of the circle in its new location?

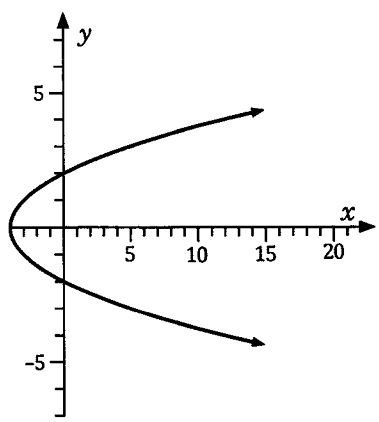
9. The circle $x^2 + y^2 - 6x + 10y + 25 = 0$ is moved left 7 units and up 2 units. What will be the equation of the circle in its new location?

10. Each of the following graphs show $y^2 = x$ translated up, down, left or right. Determine the equation of each relationship shown.

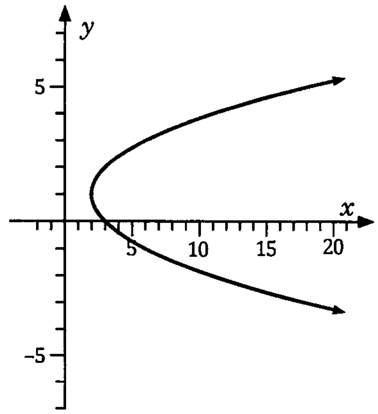
(a)



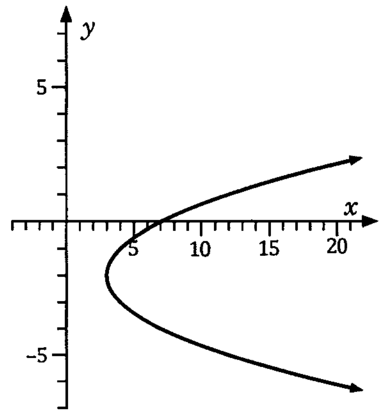
(b)



(c)



(d)



11. Point A is the centre of the circle $(x - 3)^2 + (y - 11)^2 = 144$.
Point B is the centre of the circle $(x - 12)^2 + (y + 1)^2 = 9$.
   (a) Determine the length of AB.
   (b) Determine whether the circles have two points in common, just one point in common or no points in common and justify your answer.

12. Point C is the centre of the circle $(x - 2)^2 + (y - 3)^2 = 9$.
Point D is the centre of the circle $(x + 2)^2 + (y - 5)^2 = 1$.
   (a) Determine the length of CD.
   (b) Determine whether the circles have two points in common, just one point in common or no points in common and justify your answer.

13. Find the coordinates of the points where the line $y = x - 3$ meets the circle
$$(x - 4)^2 + (y - 2)^2 = 25.$$

14. Find the coordinates of the points where the line $4y = x + 30$ meets the circle
$$(x + 5)^2 + (y - 2)^2 = 34.$$

15. Prove that the straight line $3y = x + 25$ is a tangent to the circle
$$(x - 7)^2 + (y - 4)^2 = 40,$$
and find the coordinates of the point of contact.

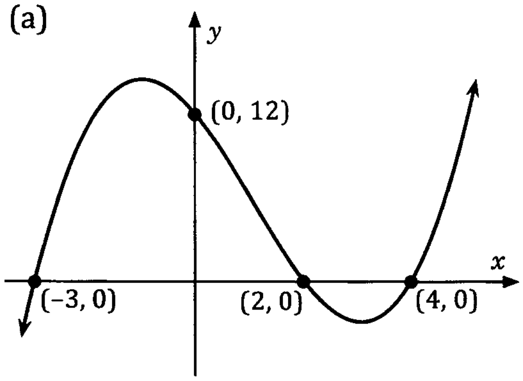
16. What restriction is there on the possible values of a if
$$x^2 + 2x + y^2 - 10y + a = 0$$
is the equation of a circle?

Miscellaneous Exercise Seven.

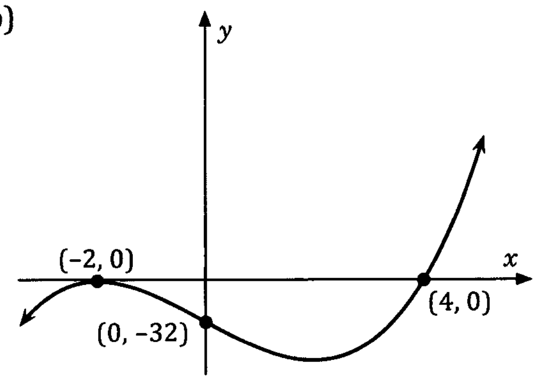
This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. Determine the equation of each of the following cubic functions.

(a)



(b)



2. Without the assistance of a calculator, solve the following quadratic equation

$$x^2 - 4x - 6 = 0,$$

(a) using the quadratic formula,
(b) by completing the square,
and in each case give answers in exact form.

3. Use the technique of completing the square to determine the centre and radius of the circle with equation: $x^2 + 6x + y^2 - 10y = 15$.

4. If $f(x) = x$, $g(x) = x^2$ and $h(x) = x^3$ determine,
(a) f(4), (b) g(4), (c) h(4)
(d) the values of p for which f(p), g(p) and h(p) are all the same.

5. State which of the following functions are linear and for those that are state the gradient.

$f_1(x)$: $y = \sqrt{x}$ $f_2(x)$: $2y = 5x + 4$

$f_3(x)$: $y = (x + 1)(x + 3)$ $f_4(x)$: $2x + y = 3$

6. Find the equation of the straight line that passes through the point (15, −1) and that is perpendicular to the line $5x + 2y = 9$.

7. Use either the graphing facility or the equation solving facility of a calculator to solve each of the following equations for $x \in \mathbb{R}$.
(a) $8x^3 + 18x^2 - 221x + 315 = 0$ (b) $8x^3 - 2x^2 = 441 + 315x$
(c) $2x^3 - 11x^2 + 19x = 12$ (d) $x^4 - 3x^3 + 12x^2 - 21x + 35 = 0$

8. For each of parts (a) to (h) below state which of the following statements apply:

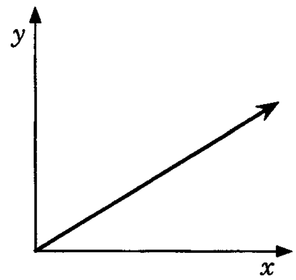
Statement A: y increases as x increases.

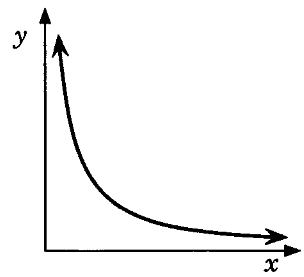
Statement B: y decreases as x increases.

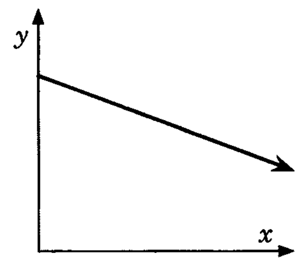
Statement C: y is directly proportional to x.

Statement D: y is inversely proportional to x.

(a) $y = 5x$ (b) $y = \frac{7}{x}$ (c) $y = \frac{2}{x}$ (d) $y = \frac{x}{3}$ (e) $y = 2x + 1$

(f)


(g)


(h)


9. Without the help of a calculator, solve each of the following equations for $x \in \mathbb{R}$.

(a) $(2x - 7)(x + 9) = 0$ (b) $x^2 - 8x + 12 = 0$

(c) $5x^2 + 2x - 3 = 0$ (d) $(x + 11)(5x - 4)(x - 7) = 0$

(e) $(x - 3)(x^2 + 4x - 5) = 0$ (f) $(x + 5)(2x^2 + x - 6) = 0$

10. Classify each of the following as appearing to be the graph of

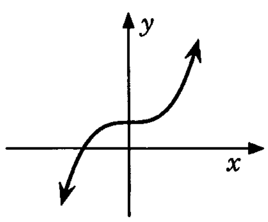
a linear function,

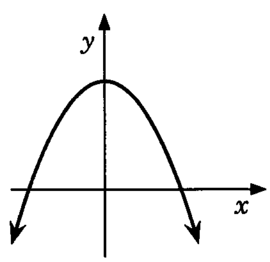
or a quadratic function,

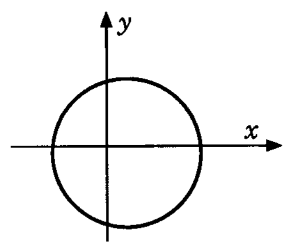
or a cubic function,

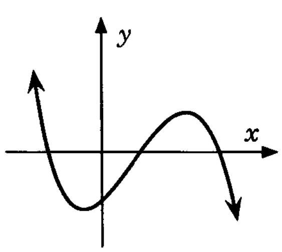
or a reciprocal function

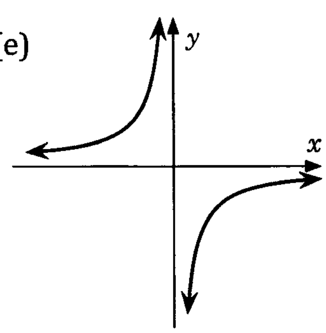
or none of the above.

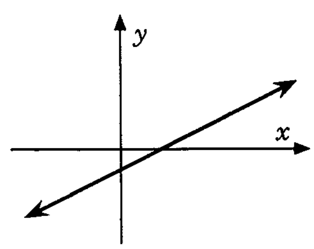
(a)


(b)


(c)


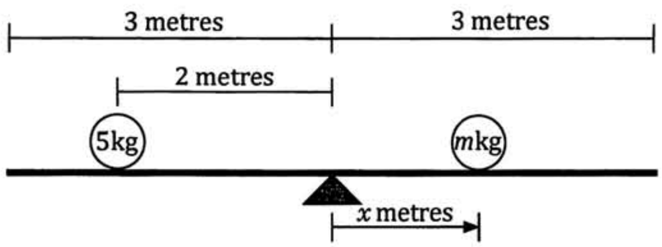
(d)


(e)


(f)


11. Given that $x^3 - 8x^2 + 19x - 12 = (x - 3)(x^2 + bx + c)$:
    (a) Determine the value of c by inspection.
    (b) With your answer from part (a) in place of c expand $(x - 3)(x^2 + bx + c)$ and hence determine b.
    (c) Factorise $x^3 - 8x^2 + 19x - 12$.

12.

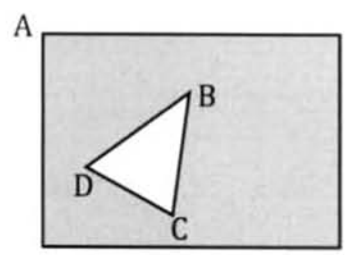


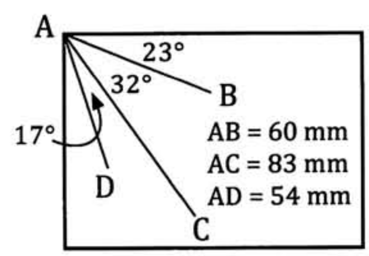
To balance the system above the relationship between m and x must be:

$$m = \frac{10}{x}$$

(a) If x is doubled in value what must happen to the value of m if the system is to remain in balance?
(b) State whether the relationship between x and m is one of direct proportion, inverse proportion or neither of these and explain what your answer means in terms of the way m needs to vary as x is varied if the system is to remain in balance.
(c) If $m = 20$ what must be the value of x for the system to be in balance?
(d) With the x values as input, the m values as output and the system in balance, what is the domain and range for this function?

13. An engineering component consists of a rectangular metal plate with a triangular piece removed, as in the diagram below left.
The removed piece is cut away by a computer controlled machine that is programmed to cut a triangle with vertices at the distances and angles shown on the diagram below right.





Find the area and the perimeter of the triangular piece that is removed.